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On a Finite-Difference Approximation for the Steady-State Navier–Stokes Equations

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An implicit type of successive approximation for the steady-state Navier–Stokes equations is proposed. The method has second-order accuracy in the whole flow field and can avoid the divergence of the iterations. As a numerical example, the flow in a square cavity is calculated for Reynolds numbers 100, 200, 400, 500, 700, and 1000. The accuracy of the convective terms is improved by the present method.

INTRODUCTION

The Navier–Stokes equations, which are considered to govern fluid motion, are very difficult to solve analytically because of their nonlinearity. Up to the present time, many workers have tried to approximate the Navier–Stokes equations by difference equations and to solve them by numerical procedures. However, because of the nonlinearity of the Navier–Stokes equations, some difficulties have arisen in numerical as well as in analytical studies. One of the greatest difficulties may be the problem of the divergence of the iterative methods at high Reynolds numbers. In past studies, many workers took an extremely small relaxation factor (for example, see Refs. [1, 2]), or used the upstream differencing method due to Greenspan [3] or Runchal *et al.* [4]. In the former method, calculations must be continued for a long time to get convergent solutions. For the upstream differences the defect is that the accuracy of convective terms is not good. Although there are some upstream differencing methods in the time-dependent problem (for example, see Refs. [5, 6]), they have only first-prder formal accuracy in the steady-state problem. There are also studies which introduce higher-order differences to convective terms (for example, see Ref. [7]). Recently, in the steady-state problem, Ozawa [8] derived a formula which can avoid the divergence of the solutions of the difference equations without reduction of the accuracy of the convective terms. His formula was derived by using Greenspan's idea and by introducing the local computing time step to the steady difference equations.

In this paper, the steady-state problem is considered. From a viewpoint of numerical analysis, a new method, different from Ozawa's, is derived. The method has second-order accuracy in the entire flow field and the divergence of finite-difference equations

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can be avoided at higher Reynolds numbers. In order to test this method, the flow in a square cavity is calculated, which is a problem already treated by others (Kawaguti [9], Burggraf [2], Pan and Acrivos [10], Greenspan [3], Runchal *et al.* [4], Bozeman and Dalton [11], Nallasamy and Prasad [12], Ozawa [8], and Nallasamy and Prasad [13]). The results of this study show the usefulness of the new method proposed here.

Formulation of A New Difference Equation

The Navier–Stokes equations for the two-dimensional steady flow of a viscous incompressible fluid can be expressed in nondimensional form as

$$\frac{\partial\psi}{\partial y}\frac{\partial\zeta}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\zeta}{\partial y} = \frac{1}{Re}\left(\frac{\partial^2\zeta}{\partial x^2} + \frac{\partial^2\zeta}{\partial y^2}\right), \tag{1}$$

using a stream function ψ and a vorticity ζ. This equation is expressed in convective form. Here, $Re = UL/\nu$ is the Reynolds number, U the characteristic speed, L the characteristic length of the flow field, and ν the kinematic viscosity of the fluid. Between ζ and ψ, there exists the relation,

$$\zeta = -\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right). \tag{2}$$

In general, Eq. (1) is transformed into a difference equation, using the centered space-differences, as

$$\frac{1}{4hs}\{(\psi_{i,j+1} - \psi_{i,j-1})(\zeta_{i+1,j} - \zeta_{i-1,j}) - (\psi_{i+1,j} - \psi_{i-1,j})(\zeta_{i,j+1} - \zeta_{i,j-1})\}$$
$$= \frac{1}{Re}\left\{\frac{1}{h^2}(\zeta_{i+1,j} + \zeta_{i-1,j} - 2\zeta_{i,j}) + \frac{1}{s^2}(\zeta_{i,j+1} + \zeta_{i,j-1} - 2\zeta_{i,j})\right\}, \tag{3}$$

where h and s are the mesh sizes in x- and y-directions respectively. That is, $\zeta_{i,j}$ is corrected successively by

$$\zeta_{i,j} = \frac{h^2 s^2}{2(h^2 + s^2)}\left(\frac{\zeta_{i+1,j} + \zeta_{i-1,j}}{h^2} + \frac{\zeta_{i,j+1} + \zeta_{i,j-1}}{s^2}\right)$$
$$+ \frac{hsRe}{8(h^2 + s^2)}\{\alpha(\zeta_{i,j+1} - \zeta_{i,j-1}) - \beta(\zeta_{i+1,j} - \zeta_{i-1,j})\}, \tag{4}$$

where $\alpha = \psi_{i+1,j} - \psi_{i-1,j}$ and $\beta = \psi_{i,j+1} - \psi_{i,j-1}$. One may use either the Jacobi or Gauss–Seidel form of Eq. (4) [17]. However, as is well known, the iterative solution of $\zeta_{i,j}$ by Eq. (4) often diverges at higher Reynolds numbers. The upstream differencing method could be introducing to avoid the divergence of the solution. However, since the upstream differenceing method contains first-order approximations in the left-hand side of Eq. (1), a false viscosity effect appears in the solution. With the

finite mesh size used, it is very doubtful that the method is an accurate approximation to Eq. (1) because of the different order of approximations in both sides of Eq. (1). Thus, the meaning of Eq. (3) must be reconsidered.

It is desirable to modify Eq. (4) to make it diagonally dominant while keeping the second-order accuracy. To this end, Eq. (3) is written as

$$\frac{1}{4hs}[(\psi_{i,j+1}-\psi_{i,j-1})\{(\zeta_{i+1,j}-\zeta_{i,j})+(\zeta_{i,j}-\zeta_{i-1,j})\} - (\psi_{i+1,j}-\psi_{i-1,j})\{(\zeta_{i,j+1}-\zeta_{i,j})+(\zeta_{i,j}-\zeta_{i,j-1})\}] = \frac{1}{Re}\left\{\frac{1}{h^2}(\zeta_{i+1,j}+\zeta_{i-1,j}-2\zeta_{i,j})+\frac{1}{s^2}(\zeta_{i,j+1}+\zeta_{i,j-1}-2\zeta_{i,j})\right\}. \quad (5)$$

Next, interpreting Eq. (3) as a successive process to the difference solution of Eq. (1), we distinguish between the kth and $(k+1)$ st approximates of the terms of $\zeta_{i,j}$, denoting them as $\zeta_{i,j}^{(k)}$ and $\zeta_{i,j}^{(k+1)}$, respectively. Then, Eq. (5) in Gauss–Seidel form becomes

$$\frac{1}{4hs}[(\psi_{i,j+1}-\psi_{i,j-1})\{(\zeta_{i+1,j}^{(k)}\mp\zeta_{i,j}^{(k+1)})+(\pm\zeta_{i,j}^{(k)}-\zeta_{i-1,j}^{(k+1)})\} - (\psi_{i+1,j}-\psi_{i-1,j})\{(\zeta_{i,j+1}^{(k)}\mp\zeta_{i,j}^{(k+1)})+(\pm\zeta_{i,j}^{(k)}-\zeta_{i,j-1}^{(k+1)})\}] = \frac{1}{Re}\left\{\frac{1}{h^2}(\zeta_{i+1,j}^{(k)}+\zeta_{i-1,j}^{(k+1)}-2\zeta_{i,j}^{(k+1)})+\frac{1}{s^2}(\zeta_{i,j+1}^{(k)}+\zeta_{i,j-1}^{(k+1)}-2\zeta_{i,j}^{(k+1)})\right\}.$$

Note that this equation denotes a new implicit type of successive approximation. In this equation, if $\alpha=\psi_{i+1,j}-\psi_{i-1,j}\geqslant 0$ and $\beta=\psi_{i,j+1}-\psi_{i,j-1}\geqslant 0$, for example, then the signs of the first two terms, $\zeta_{i,j}^{(k+1)}$ and $\zeta_{i,j}^{(k)}$, should be chosen $+$ and $-$, respectively, and the signs of the next two terms $\zeta_{i,j}^{(k+1)}$ and $\zeta_{i,j}^{(k)}$ should be chosen $-$ and $+$, respectively. In other words, the signs should be chosen so that the coefficient of $\zeta_{i,j}^{(k+1)}$ of the following Eq. (6) becomes as large as possible. Thus,

$$\left\{2\left(\frac{s}{h}+\frac{h}{s}\right)+\frac{Re}{4}(|\alpha|+|\beta|)\right\}\zeta_{i,j}^{(k+1)} = \frac{s}{h}(\zeta_{i+1,j}^{(k)}+\zeta_{i-1,j}^{(k+1)})+\frac{h}{s}(\zeta_{i,j+1}^{(k)}+\zeta_{i,j-1}^{(k+1)}) + \frac{Re}{4}\{\alpha(\zeta_{i,j+1}^{(k)}-\zeta_{i,j-1}^{(k+1)})-\beta(\zeta_{i+1,j}^{(k)}-\zeta_{i-1,j}^{(k+1)})+(|\alpha|+|\beta|)\zeta_{i,j}^{(k)}\}. \quad (6)$$

Hence, from a viewpoint of numerical analysis, a new method is derived. In the present method, if the necessary condition for convergence

$$|\zeta_{i,j}^{(k+1)}-\zeta_{i,j}^{(k)}|\to 0 \quad (7)$$

is satisfied, the truncation errors of Eq. (6) become $O(\delta^2)$ ($\delta=\max(h,s)$). Furthermore, since the terms which involve the Reynolds number as a multiplier appear on

both sides of Eq. (6), the correction of $\zeta_{i,j}^{(k+1)}$ is smaller than that of Eq. (4) at higher Reynolds numbers. Hence, the divergence of difference equations can be avoided at higher Reynolds numbers. This form of calculation is rather simple compared with those of Greenspan [3] or Runchal *et al.* [4]. The only essential weak point of this method is that it is necessary to check the value of

$$Res = Re\{(|\alpha| + |\beta|)\,|\,\zeta_{i,j}^{(k+1)} - \zeta_{i,j}^{(k)}\,|\}_{\max}/4 \tag{8}$$

to determine the accuracy of the difference equation. If this value is large, the convergent solution is not an accurate second-order approximation to Eq. (1). However, this value is small enough as will be shown later for the cavity flow problem.

Note that the difference between Eq. (6) and SOR [17] is that Eq. (6) has $Re\,(|\alpha| + |\beta|)/4$ in two places where SOR would have $2(1 - \omega)(s^2 + h^2)/(hs\omega)$. Thus, Eq. (6) may be viewed as an SOR method with variable ω.

Numerical Example

As a numerical example, the steady flow in a two-dimensional square cavity is considered (see Fig. 1). This prototype problem was first discussed by Kawaguti [9], and later by many others (see Refs. [2–4, 8, 10–13]). The coordinates of the mesh points (i,j) are $((i-1)h, (j-1)h)$, and a uniform square mesh of side h is used. Calculations are made using three methods, viz., Kawaguti's method [9] (put $h = s$ in Eq. (4)), Greenspan's method [3], and the method proposed in this paper (put $h = s$ in Eq. (6)).

In Kawaguti's method, the difference equation for $\zeta_{i,j}$ is

$$\begin{aligned}\zeta_{i,j}^{*} = {} & \tfrac{1}{4}(\zeta_{i+1,j}^{(k)} + \zeta_{i-1,j}^{(k+1)} + \zeta_{i,j+1}^{(k)} + \zeta_{i,j-1}^{(k+1)}) \\ & + (Re/16)\{\alpha(\zeta_{i,j+1}^{(k)} - \zeta_{i,j-1}^{(k+1)}) - \beta(\zeta_{i+1,j}^{(k)} - \zeta_{i-1,j}^{(k+1)})\},\end{aligned} \tag{9}$$

where $\alpha = \psi_{i+1,j} - \psi_{i-1,j}$ and $\beta = \psi_{i,j+1} - \psi_{i,j-1}$.

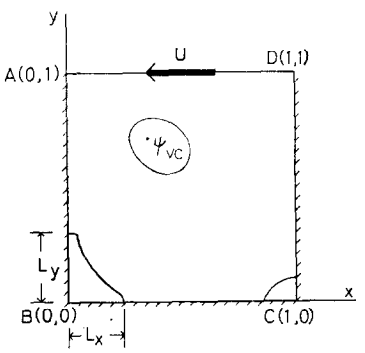

FIG. 1. Cavity flow problem.

According to Greenspan's method, the difference equation is

$$\{4 + (Re/2)(|\alpha| + |\beta|)\}\,\zeta^*_{i,j} = \zeta^{(k)}_{i+1,j} + \zeta^{(k+1)}_{i-1,j} + \zeta^{(k)}_{i,j+1} + \zeta^{(k+1)}_{i,j-1} + (Re/4)\{\alpha(\zeta^{(k)}_{i,j+1} - \zeta^{(k+1)}_{i,j-1}) + |\alpha|(\zeta^{(k)}_{i,j+1} + \zeta^{(k+1)}_{i,j-1}) - \beta(\zeta^{(k)}_{i+1,j} - \zeta^{(k+1)}_{i-1,j}) + |\beta|(\zeta^{(k)}_{i+1,j} + \zeta^{(k+1)}_{i-1,j})\}. \tag{10}$$

According to the present method, the difference equation is

$$\{4 + (Re/4)(|\alpha| + |\beta|)\}\,\zeta^*_{i,j} = \zeta^{(k)}_{i+1,j} + \zeta^{(k+1)}_{i-1,j} + \zeta^{(k)}_{i,j+1} + \zeta^{(k+1)}_{i,j-1} + (Re/4)\{\alpha(\zeta^{(k)}_{i,j+1} - \zeta^{(k+1)}_{i,j-1}) - \beta(\zeta^{(k)}_{i+1,j} - \zeta^{(k+1)}_{i-1,j}) + (|\alpha| + |\beta|)\,\zeta^{(k)}_{i,j}\}. \tag{11}$$

The stream function is corrected by

$$\psi^*_{i,j} = (\psi^{(k)}_{i+1,j} + \psi^{(k+1)}_{i-1,j} + \psi^{(k)}_{i,j+1} + \psi^{(k+1)}_{i,j-1} + h^2\zeta^{(k)}_{i,j})/4. \tag{12}$$

Equation (12) is common to the three methods. Relaxation factors are introduced as follows:

$$\zeta^{(k+1)}_{i,j} = \zeta^{(k)}_{i,j} + \omega_z(\zeta^*_{i,j} - \zeta^{(k)}_{i,j}), \tag{13}$$

where ω_z is the relaxation factor for ζ, and

$$\psi^{(k+1)}_{i,j} = \psi^{(k)}_{i,j} + \omega_p(\psi^*_{i,j} - \psi^{(k)}_{i,j}), \tag{14}$$

where ω_p is the relaxation factor for ψ. In this study, the relaxation factors ω_z and ω_p are both put equal to 1 in order to see the nature of Eq. (11). The boundary conditions are

$$\begin{aligned} &\psi = 0, \quad \partial\psi/\partial x = 0 \quad &&\text{on } AB;\\ &\psi = 0, \quad \partial\psi/\partial y = 0 \quad &&\text{on } BC;\\ &\psi = 0, \quad \partial\psi/\partial x = 0 \quad &&\text{on } CD;\\ &\psi = 0, \quad \partial\psi/\partial y = -1 \quad &&\text{on } DA. \end{aligned} \tag{15}$$

These equations can be expressed as difference equations by

$$\begin{aligned} &\psi_{1,j} = 0, \quad \zeta_{1,j} = -2\psi_{2,j}/h^2 \quad &&\text{for } 2 \leqslant j \leqslant jc - 1;\\ &\psi_{i,1} = 0, \quad \zeta_{i,1} = -2\psi_{i,2}/h^2 \quad &&\text{for } 2 \leqslant i \leqslant ic - 1;\\ &\psi_{ic,j} = 0, \quad \zeta_{ic,j} = -2\psi_{ic-1,j}/h^2 \quad &&\text{for } 2 \leqslant j \leqslant jc - 1;\\ &\psi_{i,jc} = 0, \quad \zeta_{i,jc} = -2(\psi_{i,jc-1} - h)/h^2 \quad &&\text{for } 2 \leqslant i \leqslant ic - 1. \end{aligned} \tag{16}$$

Numerical integrations of Eqs. (1) and (2) are carried out by iterative procedures according to the following steps:

(1) Initial values of $\zeta_{i,j}$ and $\psi_{i,j}$ are assumed at all mesh points.

(2) Calculate the values of vorticity on the boundary by Eq. (16).

(3) Successively calculate for every i from $i = 2$ to $ic - 1$.

(i) Correct the values of vorticity and stream function at the inner points from $j = 2$ to $jc - 1$ for fixed i ($ic = jc$).

(ii) Correct $\zeta_{i,j}$ (by Eq. (13)).

(iii) Correct $\psi_{i,j}$ (by Eq. (14)).

(4) Except for all points $\zeta_{i,j} = \psi_{i,j} \equiv 0$, $|(f_{i,j}^{(k)} - f_{i,j}^{(k-1)})/f_{i,j}^{(k)}|_{\max} \leqslant \epsilon$ ($f \equiv \zeta$ or ψ) is calculated, and if this relation is satisfied ($\epsilon = 10^{-5}$), then the iteration is terminated, k being the number of iterations. If this relation is not satisfied, then return to the step (2).

Results and Discussions

First of all, calculations are performed under the conditions that all initial values of $\zeta_{i,j}$ and $\psi_{i,j}$ are equal to zero at all mesh points for every method and the relaxation factors are taken as $\omega_z = \omega_p = 1$ in Eqs. (13) and (14). The number of iterations of these calculations is shown in Table I. In this table, a dash indicates that the calculation was not attempted, and "div." indicates that the solution of difference equations has diverged. If at least one value of $\zeta_{i,j}$ or $\psi_{i,j}$ has become larger than 10^6, then the

TABLE I

Comparison of the Number of Iterations by Various Methods ($\epsilon = 10^{-5}$)

Method	Mesh size h	Reynolds number					
		100	200	400	500	700	1000
Kawaguti	1/10	146	div.	—	—	—	—
	1/20	441	div.	—	—	—	—
	1/40	1984	2218	div.	div.	—	—
Greenspan	1/10	139	129	—	—	—	—
	1/20	460	440	566	537	619	625
	1/40	2096	1773	2499	2538	4455	6350
Present	1/10	148	130	—	—	—	—
	1/20	467	472	1291	2328	572	oscil.
	1/40	1984	2091	2572	3057	3370	4438
	1/60	—	—	—	2931	3265	3562

solution of the difference equations has been regarded as diverged. And "oscil." denotes that the solution of the difference equations has oscillated. In this case, a more detailed examination showed that $| \zeta_{i,j}^{(k)} - \zeta_{i,j}^{(k-1)} |_{\max}$ was of the order of 10 after 50 and 100 iterations, and this did not change after 500 and 1000 iterations.

For $h = 1/60$, initial values are set equal to zero at all mesh points, but the relaxation factors $\omega_z = 1.0$ and $\omega_p = 1.7$ are taken. Lastly, a finer mesh size ($h = 1/100$) is taken at $Re = 1000$. In this case, the solution of Kawaguti's method for $h = 1/50$ at $Re = 100$ is used as the initial values for $\zeta_{i,j}$ and $\psi_{i,j}$, and the relaxation factors $\omega_z = 1.0$ and $\omega_p = 1.7$ are taken. Under these conditions, 8000 iterations are performed until ϵ becomes 1.792×10^{-4}.

The necessary CPU times are 0.430 (sec/iteration) by Kawaguti's method, 0.592 by Greenspan's method, and 0.444 by the present method, at $Re = 100$ and for $h = 1/40$.

Figures 2a–e and Fig. 3 show the typical results of stream functions and vorticity. Figure 2a shows the streamlines at $Re = 200$ for $h = 1/40$ by the present method. The maximum value of $\psi_{i,j}$ is 0.1032 and is expressed as ψ_{vc} in the legends of this figure. (x_{vc}, y_{vc}) denotes the coordinate of ψ_{vc}, and ζ_{vc} is the value of $\zeta_{i,j}$ at (x_{vc}, y_{vc}). These values are defined at mesh points but not interpolated. The results of the exact second-order method (Kawaguti's method) coincide with the results of the present method in all cases where both converge. At $Re = 200$, $h = 1/40$, $\psi_{vc} = 0.1013$, $\zeta_{vc} = 2.841$, and $(x_{vc}, y_{vc}) = (0.375, 0.725)$ are the values obtained by Greenspan's method. From the result, we could say the following. At this Reynolds number, the false viscosity effect in Greenspan's method does not influence the flow field as much. Torrance [14] already pointed out the fact that, at lower Reynolds numbers, different finite-difference equations give similar results. Figures 2b and c show the streamlines by the present method at $Re = 400$ and 700 for $h = 1/40$ and $1/60$, respectively. In Greenspan's method, at $Re = 400$, $\psi_{vc} = 0.0939$, $\zeta_{vc} = 2.448$, and $(x_{vc}, y_{vc}) = (0.375, 0.700)$ are obtained for $h = 1/40$, and at $Re = 700$, $\psi_{vc} = 0.0819$, $\zeta_{vc} = 2.274$, and $(x_{vc}, y_{vc}) = (0.375, 0.700)$ for $h = 1/40$. (The values which are shown in the legend of Fig. 2c are all using $h = 1/60$; for $h = 1/40$, they are $\psi_{vc} = 0.0938$, $\zeta_{vc} = 1.827$, and $(x_{vc}, y_{vc}) = (0.450. 0.600)$ in the present method.) Thus, as the Reynolds number becomes large, there appear differences between the solutions of Greenspan's method and the present method, and Kawaguti's method diverges (see Table I). Figure 2d shows a more realistic solution with the present method for $h = 1/40$ at $Re = 1000$. At $Re = 1000$, the false viscosity effect appeared in the solution with Greenspan's method for $h = 1/40$, and an unrealistic solution was obtained with the present method for $h = 1/20$ ($\omega_z = 0.8$, $\omega_p = 1.0$). Figure 2e shows the most accurate solution ($h = 1/100$) at $Re = 1000$. We can see the second separated streamline ($L_x = 0.0141$, $L_y = 0.0140$) like Moffatt's flow in the figure.

For $h = 1/40$ and $1/100$ at $Re = 1000$, the values of the local convergence Res (see Eq. (8)) are 4.56×10^{-6} and 6.079×10^{-6}, respectively. The present difference equation is reduced to a system of algebraic equations

$$A\zeta = \mathbf{y},$$

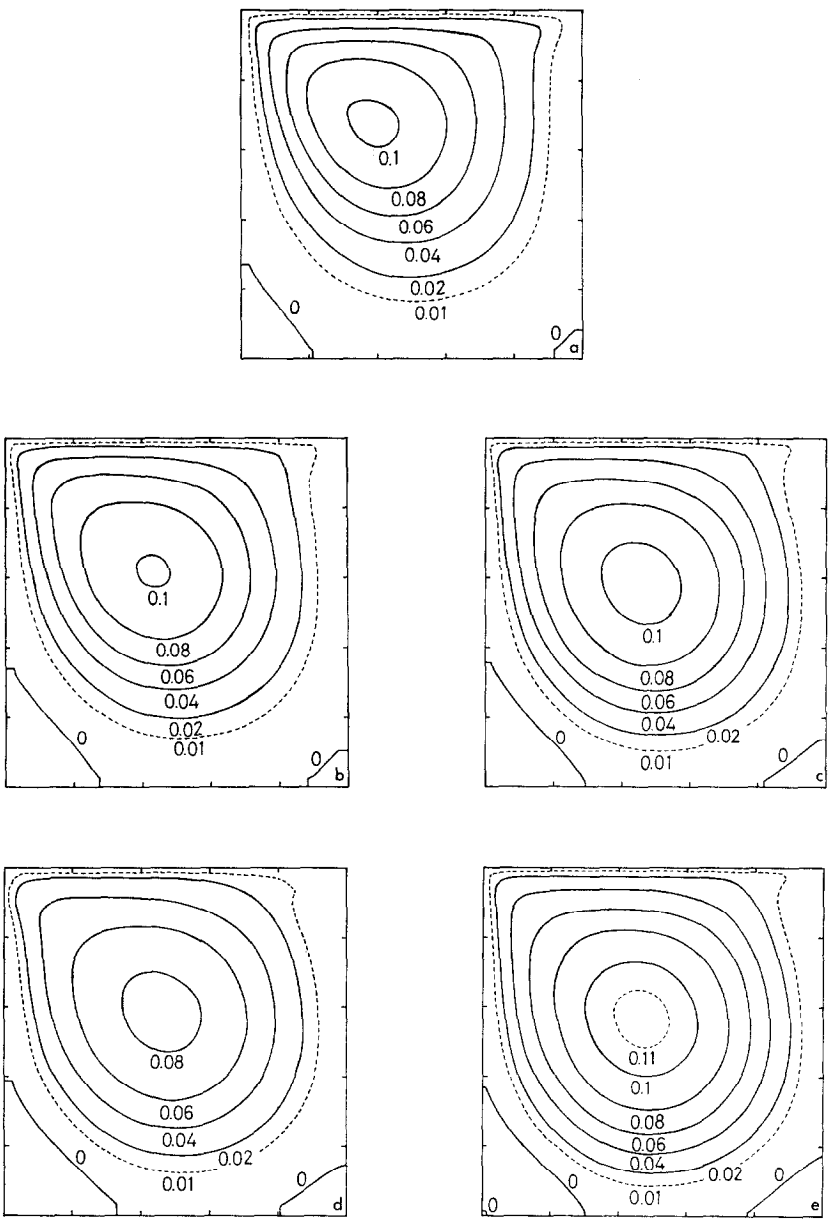

FIG. 2. Streamlines. (a) Re = 200, $h = 1/40$, present method. $(x_{vc}, y_{vc}) = (0.400, 0.675)$ $\psi_{vc} = 0.1032$, $\zeta_{vc} = 2.609$, $L_x = 0.2102$, $L_y = 0.2702$. (b) Re = 400, $h = 1/40$, present method. $(x_{vc}, y_{vc}) = (0.425, 0.625)$, $\psi_{vc} = 0.1012$, $\zeta_{vc} = 2.143$, $L_x = 0.2752$, $L_y = 0.3364$. (c) Re = 700, $h = 1/60$, present method. $(x_{vc}, y_{vc}) = (0.450, 0.583)$, $\psi_{vc} = 0.1067$, $\zeta_{vc} = 1.989$, $L_x = 0.2934$, $L_y = 0.3557$. (d) Re = 1000, $h = 1/40$, present method. $(x_{vc}, y_{vc}) = (0.450, 0.600)$, $\psi_{vc} = 0.0852$, $\zeta_{vc} = 1.614$, $L_x = 0.3294$, $L_y = 0.3873$. (e) Re = 1000, $h = 1/100$, present method. $(x_{vc}, y_{vc}) = (0.470, 0.570)$, $\psi_{vc} = 0.1133$, $\zeta_{vc} = 1.988$, $L_x = 0.3040$, $L_y = 0.3672$.

where the coefficients of the matrix A depend on the stream function, i.e., $A = A(\psi)$. The stream function is obtained from Eq, (2). Thus, the global convergence of the iterative solution to Eq. (1) is checked by

$$|(A\zeta - \mathbf{y})|/|\mathbf{y}|.$$

For $h = 1/40$ and $1/100$ at $Re = 1000$, the values of the global convergence are both $O(10^{-7})$. So, the present scheme of calculation is considered an exact second-order approximation to Eqs. (1) and (2). (With the present method, the values of the local and global convergence were $O(10^{-6}) \sim O(10^{-7})$ in all cases.)

Figure 3 shows the typical equivorticity lines at $Re = 1000$. As Burggraf [2] already stated, convective terms have begun to dominate the flow, producing a core of nearly uniform vorticity at the higher Reynolds number (see Batchelor [15]). The coordinates of the separation point L_y are compared with experimental results in Fig. 4. It is shown that the calculated size of upstream corner vortex (present method) becomes large or remain nearly constant as the Reynolds number becomes large. Ozawa's numerical results [8] shows the same tendency, but they are not shown in the figure.

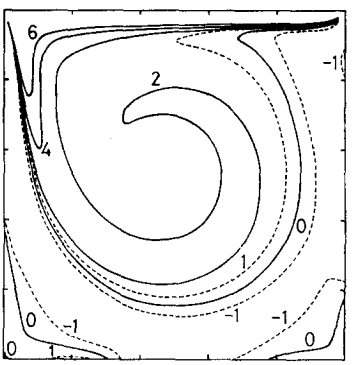

FIG. 3. Vorticity, Re = 1000, $h = 1/100$, present method.

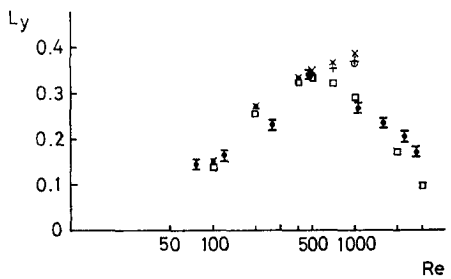

FIG. 4. Size of the upstream corner vortex of a square cavity vs Reynolds number. Experimental results: $\overline{\underline{\mathrm{T}}}$, Pan and Acrivos [10]. Numerical results: ●, Burggraf [2]; □, Nallasamy and Prasad [13]; ×, $h = 1/40$, present method; +, $h = 1/60$, present method; ○, $h = 1/100$, present method.

This tendency is also different from those of experimental results by Pan and Acrivos [10] and numerical results by Nallasamy and Prasad [13] or Bozeman and Dalton [11]. The discrepancies between these results are due to the following reasons. In the experimental results by Pan and Acrivos [10], fluid is driven by a rotating wheel on the top of a cavity. This means that the boundary conditions (15) are not realized correctly because of the curvature of the wheel. Then the results at higher Reynolds numbers are considered incorrect. The numerical results of Nallasamy and Prasad [13] and Bozeman and Dalton [11] were derived by the upstream differencing equation, the former being equivalent to Eq. (10) and the latter being a divergence form from Eq. (1). However, at the higher Reynolds number, the flow pattern is strongly dominated by the accuracy of the convective term. Thus, the results presented here are considered to be closer to the exact solution than those of [11, 13] even when the mesh size is 1/40.

Figure 5 shows the effect of the Reynolds number, the mesh size, and the accuracy of the difference equation on the location of the vortex center. From the figure, it is also found that the accuracy of finite-difference equation plays an important role for large Reynolds number and the mesh size has a strong influence on the location of the vortex center as well as on the entire flow field. As the Reynolds number becomes large, it seems that the vortex center does not approach the geometric center of the square cavity because of the growth of the second (or the third) eddy in the bottom of the cavity.

From the table and figures, we conclude the following. The proposed method in this paper is capable of avoiding the divergence of difference solutions, and the truncation errors of both sides of Eq. (1) become $O(\delta^2)$ ($\delta = \max(h, s)$) when the solution has converged. Furthermore, the number of iterations of the present method is not too different from those of other methods. Figure 6 shows the velocity distribution in the x-direction on the vertical center line of the square cavity for some Reynolds

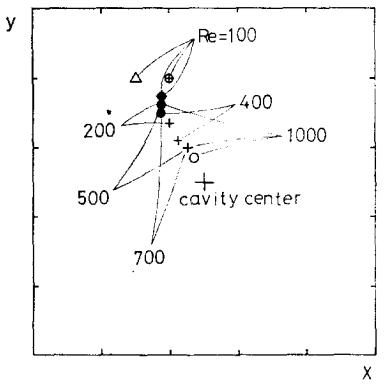

FIG. 5. Center of the center vortex: effect of Reynolds number, mesh size and accuracy of the difference equation. △, $h = 1/10$, Present method; +, $h = 1/40$, present method; ○, $h = 1/100$, present method; ⊕, $h = 1/10$, Greenspan's method; ●, $h = 1/40$, Greenspan's method.

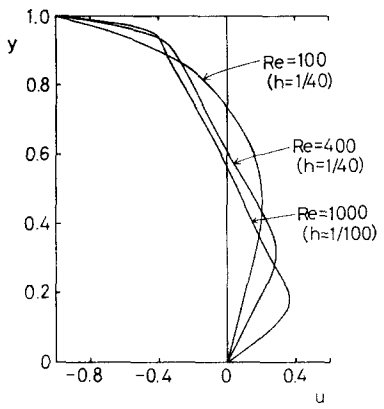

FIG. 6. Velocity profiles on vertical center line of a square cavity.

numbers. From the figure, we can see that the velocity gradient becomes more abrupt in the neighborhood of the sliding edge as the Reynolds number becomes large. This fact suggests the development of the boundary layer near the sliding edge.

CONCLUSIONS

From the viewpoint of numerical analysis, a new finite-difference approximation to the Navier–Stokes equations is derived. Using this method, some numerical experiments are performed. As a result of this study, we conclude the following:

(1) The new method proposed in this paper is useful in avoiding the divergence of the iterative method, and when the difference solutions have converged, the order of approximation is of $O(\delta^2)$ (δ: maximum mesh size) in the entire flow field.

(2) The equation to calculate vorticity is rather simple compared with that of Greenspan or Runchal *et al.*

APPENDIX: SOME REMARKS ON THIS METHOD

With the proposed method of successive approximation, the accuracy of the convective terms, which was the defect of the methods of Greenspan [3] and Runchal *et al.* [4], is improved. Recently, Ozawa [8] has derived a finite-difference equation which is different from the one proposed here. Ozawa has succeeded in obtaining an effect similar to that found in this paper. We examine the difference between these two methods. For simplicity, $\omega_z = 1$ is taken, and rewriting Eq. (11) formally, we have

$$\begin{aligned}\{4 + (Re/4)(|\alpha| + |\beta|)\}(\zeta_{i,j}^{(k+1)} - \zeta_{i,j}^{(k)}) \\ = \zeta_{i+1,j} + \zeta_{i-1,j} + \zeta_{i,j+1} + \zeta_{i,j-1} - 4\zeta_{i,j}^{(k)} \\ + (Re/4)\{\alpha(\zeta_{i,j+1} - \zeta_{i,j-1}) - \beta(\zeta_{i+1,j} - \zeta_{i-1,j})\}.\end{aligned}$$

Writing this in a generalized form gives

$$\{\lambda + (Re/\mu)(|\alpha| + |\beta|)\}(\zeta_{i,j}^{(k+1)} - \zeta_{i,j}^{(k)}) = \zeta_{i+1,j} + \zeta_{i-1,j} + \zeta_{i,j+1} + \zeta_{i,j-1} - 4\zeta_{i,j}^{(k)} + (Re/4)\{\alpha(\zeta_{i,j+1} - \zeta_{i,j-1}) - \beta(\zeta_{i+1,j} - \zeta_{i-1,j})\}. \quad \text{(A1)}$$

In this expression, λ and μ are considered constants, although they would be considered functions of k in general. We rewrite Eq. (A1) as

$$\zeta_{i,j}^{(k+1)} = \zeta_{i,j}^{(k)} + \Delta t^{-1}[\zeta_{i+1,j}^{(k)} + \zeta_{i-1,j}^{(k+1)} + \zeta_{i,j+1}^{(k)} + \zeta_{i,j-1}^{(k+1)} - 4\zeta_{i,j}^{(k)} + (Re/4)\{\alpha(\zeta_{i,j+1}^{(k)} - \zeta_{i,j-1}^{(k+1)}) - \beta(\zeta_{i+1,j}^{(k)} - \zeta_{i-1,j}^{(k+1)})\}], \quad \text{(A2)}$$

where $\Delta t = \lambda + (Re/\mu)(|\alpha| + |\beta|)$, and with $\lambda = 4$ and $\mu = 2$. This expression is very similar to that of Ozawa [8], but the iteration is done differently.

Next, we rewrite Eq. (A1) as

$$\{\lambda + (Re/\mu)(|\alpha| + |\beta|)\}\,\zeta_{i,j}^{(k+1)} = \zeta_{i+1,j}^{(k)} + \zeta_{i-1,j}^{(k+1)} + \zeta_{i,j+1}^{(k)} + \zeta_{i,j-1}^{(k+1)} + (Re/4)\{\alpha(\zeta_{i,j+1}^{(k)} - \zeta_{i,j-1}^{(k+1)}) - \beta(\zeta_{i+1,j}^{(k)} - \zeta_{i-1,j}^{(k+1)})\} + \{(\lambda - 4) + (Re/\mu)(|\alpha| + |\beta|)\}\,\zeta_{i,j}^{(k)}. \quad \text{(A3)}$$

To examine the numerical stability of Eq. (A3), we apply Wilkinson's method [16] Wilkinson has shown that the Gaussian elimination for the diagonally dominant and tridiagonal coefficient matrix is extremely stable with respect to the growth of rounding errors. Equation (A3) can be rewritten as a matrix form, of which diagonal submatrices are tridiagonal. The components of the tridiagonal matrix are, for example,

$$-\left(1 - \frac{Re}{4}\alpha\right), \qquad \lambda + \frac{Re}{\mu}(|\alpha| + |\beta|), \qquad -\left(1 + \frac{Re}{4}\alpha\right).$$

Thus, it is easily seen that there is no loss of the diagonal dominance for the larger λ and the smaller μ. So, the iterative method is more stable for the larger λ and the smaller μ. On the other hand, the left-hand side of Eq. (A1) denotes the remaining error of the Navier–Stokes equations with central-difference approximation. Therefore, if $\zeta_{i,j}^{(k+1)} - \zeta_{i,j}^{(k)}$ does not depend on the selection of parameters λ and μ, the choice of the larger λ and the smaller μ causes the number of iterations needed to converge to increase because of the magnitude of the remaining error. However, in actual calculations, $\zeta_{i,j}^{(k+1)} - \zeta_{i,j}^{(k)}$ is a function of λ, μ, and k. At present, the optimum values of λ and μ are unknown. The selection of λ and μ should be based upon the numerical stability and the asymptotic rate of convergence (see, for example, Ref. [17]), etc. The selection of $\lambda = \mu = 4$ in this paper is thought to be reasonable.

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